Preprint typeset in JHEP style - HYPER VERSION

YITP-05-34
hep-th/0506192

An invariant approach to dynamical fuzzy spaces with a three-index variable

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Abstract: A dynamical fuzzy space might be described by a three-index variable ${C_{ab}}^c$, which determines the algebraic relations $f_a f_b = {C_{ab}}^c f_c$ among the functions f_a on the fuzzy space. A fuzzy analogue of the general coordinate transformation would be given by the general linear transformation on f_a. I study equations for the three-index variable invariant under the general linear transformation, and show that the solutions can be generally constructed from the invariant tensors of Lie groups. As specific examples, I study SO(3) symmetric solutions, and discuss the construction of a scalar field theory on a fuzzy two-sphere within this framework.

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1. Introduction and a general proposal

Several thought experiments in semi-classical quantum gravity and string theory [1]-[7] suggest that the classical notion of space-time in general relativity should be replaced with a quantum one in quantum treatment of space-time dynamics. An interesting candidate is given by the non-commutative geometry or the fuzzy space [8]-[10]. The central principle of general relativity is the invariance under the general coordinate transformation. It would be obviously interesting if a fuzzy space can be obtained from formulation invariant under a fuzzy analog of the general coordinate transformation. This kind of formulation would lead to fuzzy general relativity, which may describe the gravitational dynamics and the evolution of the universe in terms of dynamical fuzzy spaces[1].

A fuzzy space is characterized by the algebraic relations $f_a f_b = C_{ab}{}^c f_c$ among the functions f_a on the fuzzy space. Therefore it would be a natural assumption that a dynamical fuzzy space can be described by treating the three-index variable $C_{ab}{}^c$ dynamically. It will be argued in the following section that a fuzzy analog of the general coordinate transformation is given by the general linear transformation on f_a. Thus the central proposal of this paper is that dynamical fuzzy spaces are described by equations for $C_{ab}{}^c$ invariant under the general linear transformation.

In the following section, a fuzzy analog of the general coordinate transformation will be discussed and the general form of the models will be proposed. In Section 3, it will be shown that the solutions to the equations can be constructed from the invariant tensors of Lie groups. In Section 4, the Lie group will be specified to SO(3), and some series of the solutions will be explicitly constructed. In Section 5, I will discuss the construction of a scalar field theory on a fuzzy two-sphere by using the result in the preceding section. The final section will be devoted to discussions and comments.

[1]The present author considered evolving fuzzy spaces in [11]-[13].

2. A fuzzy general coordinate transformation and the general form of the models

Let me first review the general coordinate transformation in the usual commutative space R^d. A basis of the continuous functions on R^d can be given by

$$\{1, x^i, x^i x^j, x^i x^j x^k, \cdots\}, \tag{2.1}$$

where x^i $(i = 1, \cdots, d)$ are the coordinates of R^d. Let f_a with an index a denote these independent functions in the set. A continuous function on R^d is given by a linear combination of f_a.

Let me consider a general coordinate transformation,

$$x'^i = f^i(x). \tag{2.2}$$

It is a natural restriction that the coordinate transformation is not singular, and is invertible. Since the right-hand side is a continuous function, the general coordinate transformation can be represented by a linear transformation,

$$x'^i = M_i{}^a f_a, \tag{2.3}$$

where $M_i{}^a$ are real.

Let me now consider a fuzzy space with a finite number of independent functions f_a $(a = 1, \cdots, n)$ on the fuzzy space. This is a fuzzy space corresponding to a compact space in usual continuous theory. I assume that all the f_a denote real functions on a fuzzy space, and that the variable $C_{ab}{}^c$, which determines the algebraic relations $f_a f_b = C_{ab}{}^c f_c$, is real. I do not assume the associativity or the commutativity of the algebra, so that $C_{ab}{}^c$ has no other constraints.

An important assumption of this paper is to interpret the transformation rule (2.3) as a partial appearance of a more general linear transformation,

$$f'_a = M_a{}^b f_b, \tag{2.4}$$

where $M_a{}^b$ can take any real values provided that the matrix $M_a{}^b$ is invertible, which comes from the assumed invertibility of the coordinate transformation (2.2). Therefore the fuzzy general coordinate transformation of this paper is given by the GL(n,R) transformation on f_a. Under the transformation (2.4), the three-index variable $C_{ab}{}^c$ transforms in the way,

$$C'_{ab}{}^c = M_a{}^{a'} M_b{}^{b'} C_{a'b'}{}^{c'} (M^{-1})_{c'}{}^c. \tag{2.5}$$

I impose that the equations of motion for $C_{ab}{}^c$ must be invariant under this GL(n,R) transformation.

For convenience, let me introduce a graphical expression. The three-index variable $C_{ab}{}^c$ can be graphically represented as in Fig. 1.

An example of an invariant equation of motion is given by

$$C_{ia}{}^{j}C_{jb}{}^{k}C_{kc}{}^{i}+C_{ai}{}^{j}C_{cj}{}^{k}C_{bk}{}^{i}=0, \tag{2.6}$$

which is graphically represented in Fig. 2. It is clear that such invariant equations of motion can be made in infinitely many ways.

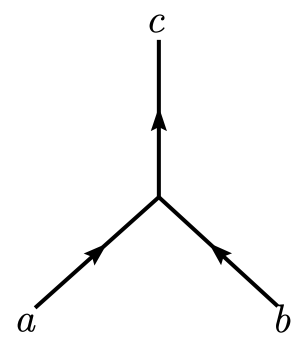

Figure 1: The graphical representation of $C_{ab}{}^{c}$. The orientation shows whether the associated index is a lower or upper one. The order of a and b can be also read in the diagram.

In the next, let me discuss the construction of an action. From the transformation property (2.5), the lower and upper indices must be contracted to make an invariant under the GL(n,R) transformation. Since an action must be invariant and $C_{ab}{}^{c}$ has more lower indices than upper, it is necessary to introduce an additional variable which has more upper indices to construct an invariant action. A way to achieve this is to define such a variable from the $C_{ab}{}^{c}$ itself. For example, one can define

$$(C_{ai}{}^{j}C_{jb}{}^{i})^{-1}, \tag{2.7}$$

which has two upper indices.

This procedure, however, will not work when the matrix in the parentheses in (2.7) is not invertible, and will also let an action be a complicated function of $C_{ab}{}^{c}$. Therefore, rather than defining a quantity like (2.7) from the beginning, I simply introduce a new variable with upper indices g^{ab} and assume it be determined from the equations of motion derived from an action. I impose its reality and the symmetry of the two indices,

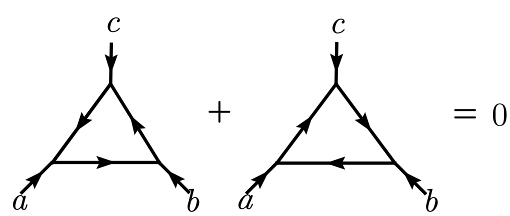

Figure 2: The graphical representation of the example equation (2.6). The oriented lines connecting vertices represent the contracted indices.

$$g^{ab}=g^{ba}. \tag{2.8}$$

The grahical representation of g^{ab} is given in Fig. 3.

Thus an invariant action is a function of the two dynamical real variables g^{ab} and $C_{ab}{}^{c}$,

$$S(g^{ab},C_{ab}{}^{c}), \tag{2.9}$$

where all the indices are contracted. The graphical representation of an invariant action is a closed diagram with oriented lines connecting blobs and three-vertices.

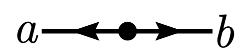

Figure 3: The graphical representation of the variable g^{ab}.

3. The classical solutions to the equations of motion

The quantum mechanical treatment of the model presented in the previous section would be obviously very interesting, but in this paper I restrict myself to the classical solutions

to the equations of motion derived from the invariant action (2.9). Let me suppose that the equations of motion for $C_{ab}{}^c$,

$$\frac{\partial S}{\partial C_{ab}{}^c}=0, \tag{3.1}$$

are satisfied at $C_{ab}{}^c=C^{0\ c}_{ab},\ g^{ab}=g_0^{ab}$. For simplicity, let me assume g_0^{ab} is invertible as a matrix. At the general values of $C_{ab}{}^c$ and g^{ab}, the invariance of the action under the GL(n,R) transformation implies

$$\frac{\partial S}{\partial g^{ab}}\delta g^{ab}+\frac{\partial S}{\partial C_{ab}{}^c}\delta C_{ab}{}^c=0, \tag{3.2}$$

where δg^{ab} and $\delta C_{ab}{}^c$ are the infinitesimally small GL(n,R) transformation of g^{ab} and $C_{ab}{}^c$,

$$\begin{aligned}\delta g^{ab}&=-\delta M_i{}^a g^{ib}-\delta M_i{}^b g^{ai},\\ \delta C_{ab}{}^c&=\delta M_a{}^i C_{ib}{}^c+\delta M_b{}^i C_{ai}{}^c-\delta M_i{}^c C_{ab}{}^i,\end{aligned} \tag{3.3}$$

where $\delta M_i{}^a$ can take any infinitesimally small real values. At the solution to (3.1), $C_{ab}{}^c=C^{0\ c}_{ab},\ g^{ab}=g_0^{ab}$, (3.2) becomes

$$\delta M_i{}^a g_0^{ib}\left.\frac{\partial S}{\partial g^{ab}}\right|_{\substack{g=g_0\\ C=C^0}}=0, \tag{3.4}$$

where I have used (2.8), (3.1) and (3.3). Therefore, since $\delta M_i{}^a$ is arbitrary and g_0^{ab} is invertible,

$$\left.\frac{\partial S}{\partial g^{ab}}\right|_{\substack{g=g_0\\ C=C^0}}=0 \tag{3.5}$$

are satisfied. This means that the equations of motion for g^{ab} are always simultaneously satisfied when the equations of motion for $C_{ab}{}^c$ are satisfied. Therefore it is enough to consider only the equations of motion (3.1) to find the classical solutions, provided g^{ab} is invertible.

Now let me consider a Lie group which has a representation of dimension n. The representation can be either reducible or irreducible. Let me embed the representation into a classical solution: The lower index of a classical solution is assumed to be transformed in the representation, while the upper one in the inverse representation. In fact, the following discussions do not depend on whether the representation is real or complex, provided that the invariant tensors considered below are real. Let g_0^{ab} be a real symmetric rank-two invariant tensor under the inverse representation. I assume the tensor g_0^{ab} is invertible as a matrix. Let me introduce $I^{\alpha\ c}_{ab}$ ($\alpha=1,2,\cdots,N$), which span all the real tensors invariant under the same transformation property as $C^{0\ c}_{ab}$. Let me assume $C^{0\ c}_{ab}$ is given by a linear combination of these invariant tensors,

$$C^{0\ c}_{ab}=A_\alpha I^{\alpha\ c}_{ab}, \tag{3.6}$$

where A_α ($\alpha=1,2,\cdots,N$) are real coefficients. Since the action is obviously invariant under the transformation of the Lie group, the left-hand side of (3.1) becomes a real invariant tensor when g^{ab} and $C_{ab}{}^c$ are substituted with the invariant tensors above. Therefore,

after exchanging the upper and lower indices by using g_0^{ab} and its inverse g^0_{ab}, the left-hand side of (3.1) can be expressed as a linear combination of $I_{ab}^{\alpha\ c}$,

$$g^0_{ai}g^0_{bj}g_0^{ck}\left.\frac{\partial S}{\partial C_{ij}{}^k}\right|_{\substack{g=g_0\\ C=C^0}} = B_\alpha(A_1, A_2, \cdots, A_N)I_{ab}^{\alpha\ c}, \tag{3.7}$$

where $B_\alpha(A_1, A_2, \cdots, A_N)$ are some functions of $A_1, A_2, \cdots, A_N$, which are determined from the specific form of the left-hand side of (3.1). Therefore the equations of motion (3.1) are reduced to the following simultaneous equations for $A_1, A_2, \cdots, A_N$,

$$B_\alpha(A_1, A_2, \cdots, A_N) = 0, \quad (\alpha = 1, \cdots, N). \tag{3.8}$$

These equations are much easier to solve than (3.1). Since the numbers of the variables and the equations are the same, the simultaneous equations (3.8) will generally have some number of solutions. The solutions are generally complex, but real solutions can be actually found in some interesting cases.

4. Explicit solutions for SO(3)

The Lie algebra of SO(3) is given by

$$[J_z, J_\pm] = \pm J_\pm, \quad [J_+, J_-] = 2J_z, \tag{4.1}$$

and an irreducible representation labeled with an integer spin j can be explicitly given by

$$\begin{aligned} J_z|j,m\rangle &= m\,|j,m\rangle, \\ J_\pm|j,m\rangle &= \sqrt{j(j+1)-m(m\pm 1)}\,|j,m\pm 1\rangle. \end{aligned} \tag{4.2}$$

It is convenient to associate the index of the model with a pair (j, m), where j is the spin, and m the J_z eigenvalue ($m = -j, -j+1, \cdots, j$). The above representation of SO(3) is generally complex but the invariant tensors can be taken real,

$$\begin{aligned} g_0^{(j_1,m_1)\,(j_2,m_2)} &= \delta_{j_1\,j_2}\delta_{m_1\,-m_2}(-1)^{m_1}, \\ I_{(j_1,m_1)\,(j_2,m_2)\,(j_3,m_3)} &= \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \end{aligned} \tag{4.3}$$

where the right-hand side in the second line is the $3j$-symbol [14, 15]. These invariant tensors are essentially unique when the spins of the representations are given.

Let me consider a representation given by the direct sum of a number of the irreducible representations. Here I consider the case that each irreducible representation appears at most once in the direct sum. Then the ansatz (3.6) becomes

$$C^0_{(j_1,m_1)\,(j_2,m_2)\,(j_3,m_3)} = A(j_1, j_2, j_3)\, I_{(j_1,m_1)\,(j_2,m_2)\,(j_3,m_3)}, \tag{4.4}$$

where the spins j_i must be contained in the representation. Since the $3j$-symbol has the cyclic symmetry $1 \to 2 \to 3 \to 1$, this is also imposed on $A(j_1, j_2, j_3)$. It is assumed that

$A(j_1,j_2,j_3)$ vanishes if j_i do not satisfy the triangle inequalities because $I_{(j_1,m_1)\,(j_2,m_2)\,(j_3,m_3)}$ vanishes identically.

The $3j$-symbol satisfies the identity,

$$\sum_{M_i}(-1)^{\sum_{i=1}^3 l_i+M_i}\begin{pmatrix} l_1 & l_2 & j_3 \\ M_1 & -M_2 & m_3\end{pmatrix}\begin{pmatrix} l_2 & l_3 & j_1 \\ M_2 & -M_3 & m_1\end{pmatrix}\begin{pmatrix} l_3 & l_1 & j_2 \\ M_3 & -M_1 & m_2\end{pmatrix} = \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3\end{Bmatrix}, \tag{4.5}$$

where $\{:::\}$ denotes the $6j$-symbol. This identity can be used to compute

$$D^0_{abc} = C^0_{ail}C^0_{bki'}C^0_{cl'k'}g_0^{ii'}g_0^{ll'}g_0^{kk'} \tag{4.6}$$

in Fig. 4, where the roman indices abbreviate the pairs (j,m). The result is

$$D^0_{(j_1,m_1)\,(j_2,m_2)\,(j_3,m_3)} = (-1)^{\sum_{i=1}^3 j_i}\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3\end{pmatrix} \times\sum_{l_i}(-1)^{\sum_{i=1}^3 l_i}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3\end{Bmatrix}A(j_1,l_3,l_2)A(j_2,l_1,l_3)A(j_3,l_2,l_1). \tag{4.7}$$

In the above derivation I have also used the property that the $3j$-symbol in (4.3) changes its sign by $(-1)^{\sum_{i=1}^3 j_i}$, when a pair of the rows are interchanged. Let me consider the equations of motion,

$$C^0_{abc} - gD^0_{abc} = 0, \tag{4.8}$$

where g is a real coupling constant. The graphical representation is given in Fig. 5. The action which leads to the equations of motion (4.8) can be constructed in several ways. This will not be discussed here, because the classical solutions are the main interest in this paper. From the result (4.7), the equations of motion are reduced to

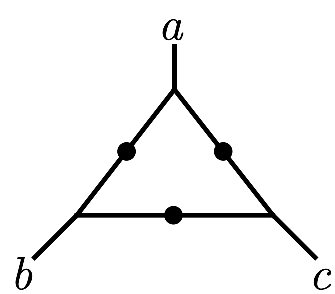

Figure 4: The graphical representation of D^0_{abc}. The arrows can be neglected, because C^0_{abc} in (4.4) has the cyclic symmetry.

$$A(j_1,j_2,j_3) - g\sum_{l_i}(-1)^{\sum_{i=1}^3 l_i+j_i}\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3\end{Bmatrix}A(j_1,l_3,l_2)A(j_2,l_1,l_3)A(j_3,l_2,l_1) = 0. \tag{4.9}$$

A series of the solutions to (4.9) parameterized by a spin parameter L can be constructed in the following way. The $6j$-symbol satisfies the identity,

$$\sum_{l_i}(-1)^{\sum_{i=1}^3 l_i}\prod_{i=1}^3(2l_i+1)\begin{Bmatrix} j_1 & j_2 & j_3 \\ l_1 & l_2 & l_3\end{Bmatrix}\begin{Bmatrix} j_1 & l_3 & l_2 \\ b & a_2 & a_3\end{Bmatrix}\begin{Bmatrix} j_2 & l_1 & l_3 \\ b & a_3 & a_1\end{Bmatrix}\begin{Bmatrix} j_3 & l_2 & l_1 \\ b & a_1 & a_2\end{Bmatrix} = (2b+1)(-1)^{b+\sum_{i=1}^3 a_i+j_i}\begin{Bmatrix} j_1 & j_2 & j_3 \\ a_1 & a_2 & a_3\end{Bmatrix}. \tag{4.10}$$

Therefore,

$$A(j_1, j_2, j_3) = \frac{1}{\sqrt{(2L+1)\, g}} \prod_{i=1}^{3} \sqrt{2j_i + 1} \begin{Bmatrix} j_1 & j_2 & j_3 \\ L & L & L \end{Bmatrix} \tag{4.11}$$

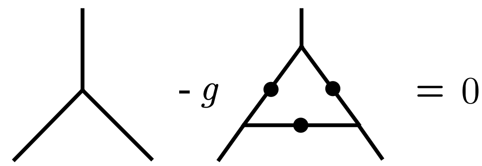

Figure 5: The equations of motion (4.8).

is the solution to (4.9) for any given spin L. Note that $\forall j_i \leq 2L$ for $A(j_1, j_2, j_3)$ to be non-zero, because the six arguments of the $6j$-symbol must be the edge lengths of a tetrahedron. Therefore the solution (4.11) can be embedded in a finite-dimensional model where the index of the model runs over (j, m) $(j = 0, 1, \cdots, 2L;\ m = -j, -j+1, \cdots, j)$.

I will now discuss another series of the solutions which will be used in the construction of a scalar field theory on a fuzzy two-sphere. The only non-vanishing components of $C^0{}_{ab}{}^c$ are assumed to be

$$C^0{}_{(1,m_1)\,(L,m_2)}{}^{(L,m_3)} = A \begin{pmatrix} 1 & L & L \\ m_1 & m_2 & -m_3 \end{pmatrix} g_0^{(L,-m_3)\,(L,m_3)}, \tag{4.12}$$

where L is a given spin and A is a real coefficient, which will be determined from the equations of motion. The rank-two symmetric tensor g_0^{ab} is as before. The index of the model runs over $(1, m)$ $(m = -1, 0, 1)$, (L, m) $(m = -L, -L+1, \cdots, L)$. The cyclic symmetry for the indices of $C^0{}_{ab}{}^c$ is not imposed this time, and $C^0{}_{ab}{}^c$ are non-zero only when its first index has a spin one and the others a spin L. The equations of motion considered are

$$C_{ab}{}^c - g\, C_{ai}{}^j C_{ki'}{}^{b'} C_{k'c'}{}^{j'} g^{ii'} g_{jj'} g^{kk'} g_{bb'} g^{cc'} = 0, \tag{4.13}$$

where g is the coupling constant and g_{ab} is the inverse of g^{ab}. Note that the invertibility of g^{ab} must be assumed from the beginning in defining this model. The graphical representation is given in Fig. 6.

An action which leads to the equations of motion (4.13) can be constructed, for example, by summing the squares of the equations of motion with appropriate contractions of the indices by g_{ab}, g^{ab}. Using the identity (4.5), it can be shown that (4.12) is actually a solution to (4.13) if

$$A = \frac{1}{\sqrt{g \begin{Bmatrix} 1 & L & L \\ 1 & L & L \end{Bmatrix}}} = \sqrt{\frac{L(L+1)(2L+1)}{g(L^2+L-1)}}. \tag{4.14}$$

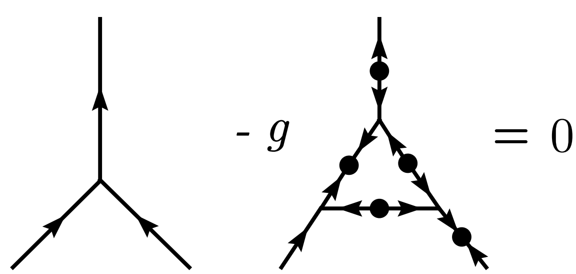

Figure 6: The equation of motion (4.13). The blobs with arrows in the inverse directions to Fig. 3 represent g_{ab}, the inverse of g^{ab}.

5. A scalar field theory on a fuzzy two-sphere

In the continuum case, the spherical harmonic functions give the independent functions on a two-sphere. They are labeled with the SO(3) indices (j, m), where the SO(3) is the rotational symmetry in the three-dimensional Euclidean space embedding the two-sphere. In the discussions in Section 2, f_a denotes the functions on a fuzzy space. Therefore it would seem reasonable to consider the solutions (4.4), (4.11) to represent the fuzzy two-sphere, because the ranges of the indices match between the spherical harmonics and the model. But I have not succeeded in this direction. The main reason for this failure is that it is difficult to obtain the spectra of the Laplacian $-j(j+1)$ from the solutions (4.4), (4.11). For example, using the properties of the $3j$- and $6j$-symbols, one finds

$$C^0_{akl} C^0_{bl'k'} g_0^{kk'} g_0^{ll'} = \frac{1}{g} g^0_{ab}, \tag{5.1}$$

where the spins of $a, b \leq 2L$. There is no dependence on j in (5.1). I have computed some other invariants, but have not found the appropriate dependence on j.

A way to construct successfully a scalar field theory on a fuzzy two-sphere can be obtained from the solutions (4.12), (4.14). This construction is almost similar to the original one [16]. In the paper [16], the kinetic term of a scalar field theory is obtained from the quadratic Casimir. The generator J_i of SO(3) in the spin L representation has two spin L indices and a spin 1. This index structure is the same as $C^0_{(1,m_1)\,(L,m_2)}{}^{(L,m_3)}$, and the kinetic term can be constructed in a similar way. An extra issue in the present model is that the indices contain $(1, m)$ as well as (L, m). It must be checked that the scalar field components with these extra indices do not destroy the wanted spectra.

Let me consider the following four-index invariant,

$$K^{a'b}_{b'a} = -C^0_{ia}{}^{a'} C^0_{i'b'}{}^{b} g_0^{ii'}, \tag{5.2}$$

shown in Fig. 7.

This invariant can be regarded as an operator on ϕ^a_b,

$$K^{a'b}_{b'a} \phi^a_b, \tag{5.3}$$

where ϕ^a_b will be identified as a scalar field on a fuzzy two-sphere.

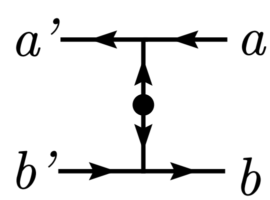

Figure 7: The quantity $K^{a'b}_{b'a}$ (5.2).

Since the only non-vanishing components of $C^0_{ab}{}^c$ are (4.12), the operator $K^{a'b}_{b'a}$ is non-trivial only on $\phi^{(L,m_1)}_{(L,m_2)}$, while it vanishes on $\phi^{(L,m_1)}_{(1,m_2)}$, $\phi^{(1,m_1)}_{(L,m_2)}$, $\phi^{(1,m_1)}_{(1,m_2)}$. From the composition rule of two spins, the scalar field $\phi^{(L,m_1)}_{(L,m_2)}$ can be decomposed into the components with total spins $J = 0, 1, \cdots, 2L$. The scalar field with a total spin J has the components,

$$(\phi_{J,m})^{(L,m_1)}_{(L,m_2)} = g_0^{(L,m_1)\,(L,-m_1)} \begin{pmatrix} L & L & J \\ -m_1 & m_2 & m \end{pmatrix}. \tag{5.4}$$

By using the following identities of the $3j$- and $6j$-symbols,

$$\sum_{h,m_h} (-1)^{m_h} (2h+1) \begin{Bmatrix} j_1 & J_1 & h \\ J_2 & j_2 & f \end{Bmatrix} \begin{pmatrix} j_1 & J_1 & h \\ m_1 & M_1 & -m_h \end{pmatrix} \begin{pmatrix} j_2 & J_2 & h \\ m_2 & M_2 & m_h \end{pmatrix}$$

$$= (-1)^{j_2+J_1} \sum_{m_f} (-1)^{m_f} \begin{pmatrix} j_1 & j_2 & f \\ m_1 & m_2 & -m_f \end{pmatrix} \begin{pmatrix} J_1 & J_2 & f \\ M_1 & M_2 & m_f \end{pmatrix}, \tag{5.5}$$

and

$$\sum_{m_1,m_2} \begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{pmatrix} j_1 & j_2 & j_3' \\ m_1 & m_2 & m_3' \end{pmatrix} = \frac{1}{2j_3+1} \delta_{j_3 j_3'} \delta_{m_3 m_3'}, \tag{5.6}$$

one can show that $\phi_{J,m}$ is an eigen vector of the operator $K_{b'a}^{a'b}$,

$$K_{b'a}^{a'b} (\phi_{J,m})^a_b = (-1)^J A^2 \begin{Bmatrix} L & L & J \\ L & L & 1 \end{Bmatrix} (\phi_{J,m})^{a'}_{b'} = \frac{J(J+1) - 2L(L+1)}{2g(L^2+L-1)} (\phi_{J,m})^{a'}_{b'}. \tag{5.7}$$

Let me define the kinetic operator $\tilde{K}$ by

$$\tilde{K}_{b'a}^{a'b} = K_{b'a}^{a'b} + \frac{L(L+1)}{g(L^2+L-1)} I_{b'a}^{a'b}, \tag{5.8}$$

where $I_{b'a}^{a'b}$ denotes the identity operator, whose graphical representation is shown in Fig. 8.

The spectra of the operator $\tilde{K}$ in the $\phi^{(L,m_2)}_{(L,m_1)}$ sector are given by $J(J+1)/2g(L^2+L-1)$ $(J = 0, 1, \cdots, 2L)$. This is the spectra of a massless free scalar field on a fuzzy two-sphere of a size $\sqrt{2g(L^2+L-1)}$. The fields in the other sectors $\phi^{(L,m_1)}_{(1,m_2)}$, $\phi^{(1,m_1)}_{(L,m_2)}$, $\phi^{(1,m_1)}_{(1,m_2)}$ obtain a mass of approximately $\sqrt{1/g}$ on account of the identity operator in (5.8). These fields components can be physically decoupled by regarding g to be small enough[2].

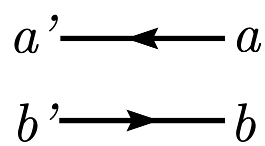

Figure 8: The identity operator $I_{b'a}^{a'b}$.

6. Discussions and comments

In this paper, I studied models with a three-index variable and discussed whether they can be used as models for dynamical fuzzy spaces. The classical solutions of the models contain many solutions with Lie group symmetries. I considered the solutions with SO(3) symmetry as specific examples, and have constructed a scalar field theory on a fuzzy two-sphere. The construction in Section 4 and 5 would be straightforwardly applied to the general SO(n) group to obtain the fuzzy complex quadratics discussed in [17].

The Lie group symmetries can be different from the orthogonal group, provided that the invariant tensors can be taken real. Moreover even in the SO(3) case, the model contains other solutions than what were used in the construction of a fuzzy-two sphere. It would be obviously interesting to find the interpretation of the other solutions as fuzzy spaces.

In the construction of a scalar field theory on a fuzzy two-sphere in Section 5, the scalar field was introduced as an additional degree of freedom, and the construction was rather adhoc. It is clear that the dynamics of ${C_{ab}}^c$ and g^{ab} is more interesting. One could analyze their quadratic fluctuations around the classical solutions. Some of the fluctuations would be identified as scalar fields, and higher-spin fields would be also found. One might suspect

[2]Since the model has no scales, only a relative scale is physically relevant.

that the models contain too many fields, but this might turn out to be nice, since it was recently argued that higher-spin fields must appear in gauge theory on non-associative fuzzy spaces [18, 19].

Actions were not explicitly given, because the main interest of this paper was the classical solution. On the other hand, actions will be needed to perform the analysis in the preceding paragraph and also to study the quantum properties of the models. The quantum process could describe the transitions between distinct fuzzy spaces with different symmetries, and is worth to study. Considering a real action with complex variables would be also interesting, since the discussions on real variables in this paper can be essentially applied also to the complex case and the solutions can be constructed more freely.

The argument about the fuzzy general coordinate transformation has remained inconsistent. In the discussions about the fuzzy general coordinate transformation in Section 2, f_a denote the functions on a fuzzy space. However, in the successful construction of the scalar field theory in Section 5, the scalar field has two indices and cannot be identified with f_a. Another formulation of field theory consistent with the argument or another interpretation of the fuzzy general coordinate transformation seems to be required.

It is shown in Section 3 that a lot of solutions can be constructed from the invariant tensors of Lie groups. A question is how many of the solutions have the Lie group symmetries. If most of the solutions do, it would be interesting to use the models as fuzzy higher dimensions [20]-[22] to explain the origin of the symmetries in our world. Incorporation of fermionic degrees of freedom and supersymmetry will be also interesting as phenomenology.

Acknowledgments

The author was supported by the Grant-in-Aid for Scientific Research No.13135213 and No.16540244 from the Ministry of Education, Science, Sports and Culture of Japan.

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